

Relativistic hydrodynamics from projection operator method

Yuki Minami and Yoshimasa Hidaka

Theoretical Research Division, Nishina Center, RIKEN, Wako 351-0198, Japan

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We study relativistic hydrodynamics in the linear regime, based on Mori's projection operator method. In relativistic hydrodynamics, it is considered that ambiguity about the fluid velocity occurs from a choice of a local rest frame: the Landau and Eckart frames. We find that the difference of the frames is not the choice of the local rest frame, but rather that of dynamic variables in the linear regime. We derive hydrodynamic equations in the both frames by the projection operator method. We show that natural derivation gives the linearized Landau equation. Also, we find that, even for the Eckart frame, the slow dynamics is actually described by the dynamic variables for the Landau frame.

I. INTRODUCTION

Relativistic hydrodynamics has been widely applied for studying relativistic nonequilibrium phenomena. For examples, it describes hadron spectra and elliptic flow in the heavy ion physics [1, 2], and jets like gamma-ray bursts in the astrophysics [3, 4]. The hydrodynamic equations applied to these systems are mainly those for perfect fluids. One of reasons is that the dissipative effects in relativistic hydrodynamics are not fully understood, e.g. some pathological problems arise from the dissipative effects: the acausal propagation and the instability of the equilibrium state [5]. Although many hydrodynamic equations have been proposed to resolve these problems [6–11], it is not still obvious which equation describes the correct behavior of the relativistic dissipative fluid. Namely, even the basic equation has not been established in relativistic hydrodynamics.

The relativistic hydrodynamic equations are generally given as the following conservation laws:

$$\partial_\mu j^\mu = 0, \quad (1)$$

$$\partial_\mu T^{\mu\nu} = 0. \quad (2)$$

Here, j^μ is the particle current and $T^{\mu\nu}$ is the energy-momentum tensor. They are decomposed to

$$j^\mu = nu^\mu + \nu^\mu, \quad (3)$$

$$T^{\mu\nu} = hu^\mu u^\nu - Pg^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \tau^{\mu\nu}, \quad (4)$$

where n is the particle density, $h = e + P$ the enthalpy density, P the pressure, e the energy density, u^μ the fluid-four velocity. The dissipative terms, ν^μ , q^μ , and $\tau^{\mu\nu}$, denote the particle and energy diffusions, and the viscous stress tensor, respectively. The explicit expressions of these terms are not unique but depending on considered equations. This ambiguity comes from a choice of local rest frames of the fluid.

To see this ambiguity, let us classify the hydrodynamic equations into two groups: the Eckart and Landau frames [12, 13]. In the Eckart frame, the local-rest frame is defined as that of the particle current, i.e., the fluid velocity is proportional to the particle current:

$$u_E^\mu \propto j^\mu. \quad (5)$$

In this frame, the particle diffusion is absent, $\nu^\mu = 0$. On the other hand, in the Landau frame, the fluid velocity is proportional to the energy current:

$$u_L^\mu \propto u_L^\nu T_\nu^\mu. \quad (6)$$

In contrast to the Eckart frame, the energy diffusion is absent, $q^\mu = 0$. We note that nonrelativistic hydrodynamics do not have these ambiguity. In the nonrelativistic limit, the energy current is identical to the particle current because the mass energy dominates the energy of fluids. Actually, the Navier-Stokes equation does not have such ambiguity and the frames. It is considered that this difference between the frames is just by the references frames and apparent. However, several differences, which are not just apparent, actually exist. For example, the Eckart frame has the instability of the equilibrium state, but the Landau frame does not.

To discuss the difference of the Landau and Eckart frames, we consider fluctuations from the global equilibrium state, namely, the linear nonequilibrium regime. The merit of this fluctuating state is that we can observe the state at the same rest frame for the energy and particle currents. We note that, at the equilibrium state, the particle and energy currents rest: $u_L^\mu = u_E^\mu = (1, \mathbf{0})$. Then, in the fluctuating state, we also have the same reference frame for the Landau and Eckart frames because the considered state is just perturbed from the equilibrium one; moreover, we need not to bother about what are *local equilibrium* and *local rest* for the relativistic system.

To see relativistic hydrodynamics in the linear regime, let us consider the Landau and Eckart equations as examples. For the Landau equation, the dissipative terms read

$$\nu^\mu = \lambda \left(\frac{nT}{h} \right)^2 \partial_\perp^\mu (\beta_\mu), \quad (7)$$

$$q^\mu = 0, \quad (8)$$

$$\tau^{\mu\nu} = \eta \left[\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} (\partial_\perp \cdot u) \right] + \zeta \Delta^{\mu\nu} (\partial_\perp \cdot u), \quad (9)$$

where λ , η and ζ are the thermal conductivity, the share and bulk viscosities, respectively. $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is a projection and $\partial_\perp^\mu \equiv \Delta^{\mu\nu} \partial_\nu$ is the space-like derivative.

Now, we linearize the Landau equation about fluctuations from the equilibrium state. Let us write $n(x) = n_0 + \delta n(x)$, $e(x) = e_0 + \delta e(x)$, $P(x) = P_0 + \delta P(x)$, $(\beta\mu)(x) = (\beta\mu)_0 + \delta(\beta\mu)(x)$, and $u^\mu(x) = u_0^\mu + \delta u^\mu(x)$. Here, the symbols with the prefix δ , denote the fluctuations. The equilibrium values are denoted by the suffix 0. Hereafter, we employ variables with the suffix and the prefix as the equilibrium values and fluctuations, respectively. For simplicity, let us choose the rest frame as the reference frame: $u_0^\mu = (1, \mathbf{0})$. Then, by the relation in the linear regime, $u_0^\mu \delta u_\mu = 0$, the fluid-velocity fluctuation is written as

$$\delta u^\mu = (0, \delta \mathbf{v}_L). \quad (10)$$

In consequence, the Landau equation is linearized as

$$\partial_0 \delta n = -n_0 \nabla \cdot \delta \mathbf{v}_L + \lambda \left(\frac{n_0 T_0}{h_0} \right)^2 \nabla^2 \delta(\beta\mu), \quad (11)$$

$$\partial_0 \delta e = -h_0 \nabla \cdot \delta \mathbf{v}_L, \quad (12)$$

$$\partial_0 (h_0 \delta \mathbf{v}_L) = -\nabla(\delta P) + \left(\zeta + \frac{1}{3} \eta \right) \nabla(\nabla \cdot \delta \mathbf{v}_L) + \eta \nabla^2 \delta \mathbf{v}_L. \quad (13)$$

Next, let us move on to the Eckart equation. the dissipative terms of the Eckart equation are

$$\nu^\mu = 0, \quad (14)$$

$$q^\mu = \lambda(\partial_\perp^\mu T - T D u^\mu), \quad (15)$$

$$\tau^{\mu\nu} = \eta \left[\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} (\partial_\perp \cdot u) \right] + \zeta \Delta^{\mu\nu} (\partial_\perp \cdot u), \quad (16)$$

where $D \equiv u^\mu \partial_\mu$ is the time-like derivative. The linearized equations are

$$\partial_0 \delta n = -n_0 \nabla \cdot \delta \mathbf{v}_E, \quad (17)$$

$$\partial_0 \delta e = -h_0 \nabla \cdot \delta \mathbf{v}_E + \lambda(\nabla^2 \delta T + T_0 \partial_0 \nabla \cdot \delta \mathbf{v}_E), \quad (18)$$

$$\partial_0 (h_0 \delta \mathbf{v}_E) = -\nabla(\delta P) + \eta \nabla^2 \delta \mathbf{v}_E + \left(\zeta + \frac{1}{3} \eta \right) \nabla(\nabla \cdot \delta \mathbf{v}_E) + \lambda \left(\nabla \partial_0 \delta T + T_0 \partial_0^2 \delta \mathbf{v}_E \right). \quad (19)$$

We note that the linearized Landau and Eckart equations have different forms even in the same rest frame, as we mentioned.

To investigate relativistic hydrodynamics in the linear regime, we here use Mori's projection operator method [15]. Mori's projection operator method is a powerful tool for extracting slow dynamics. This method is widely applied and succeed in condensed matter physics [16–18]. Actually, various slow dynamics, e.g. the Navier-Stokes, Langevin, Boltzmann equations, and equations for Nambu-Goldstone bosons, are derived [17, 19, 20]. The merit of the projection operator method is that we can derive slow dynamics only by choosing slow variables and commutation relations of those without microscopic details. We note that dynamics on macroscopic scale can be described by much fewer degrees of freedom than those on microscopic scale. Such degrees of freedom are called the slow variables (or gross variables). The slow variables are degrees of freedom that label a macroscopic state and describe long-time behavior.

The view point of the projection operator method tells us that a choice of the slow variables is a key for hydrodynamics. From the linearized Landau and Eckart equations, (11)-(13) and (17)-(19), we see that the dynamic variables are given as the energy and particle densities, and fluid-velocity fluctuations [33]:

$$\{\delta e, \delta n, \delta v_{L,E}^i\}. \quad (20)$$

The fluid velocities are different depending on the frames:

$$\delta v_E^i = n_0^{-1} j^i, \quad (21)$$

$$\delta v_L^i = h_0^{-1} T^{0i}. \quad (22)$$

Namely, the difference of the frames is the choices of the slow variables.

The important point about the choices is that j^i is not essentially slow because it is not a conserved charge. In contrast, T^{0i} is a conserved charge and slow because the energy current is equivalent to the momentum for the relativistic system: $T^{0i} = T^{i0}$. Actually, we shall find that a slow part in j^i originates from T^{0i} . We will provide

detail on this point in Sec. II C and V. Then, in this paper, we will apply the projection operator method for the choices,

$$\{\delta e, \delta p^i, \delta n\}, \quad (23)$$

$$\{\delta e, \delta p^i, \delta n, \delta j^i\}, \quad (24)$$

where $\delta p^i \equiv T^{0i}$ and $\delta j^i \equiv j^i$. We will show that the first set of slow variables, Eq. (23), gives the linearized Landau equations (11)-(13). Furthermore, we will derive the equations for the second set, Eq. (24), which include the Landau and Eckart frames. Then, we will derive the linearized Eckart equations by eliminating δp^i from the equations for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$. We note that, to correctly treat the slow part of δj^i coming from δp^i , we have to choose the both of them as the slow variables.

After that, we will discuss properties of the derived equations. We shall show that the equations for $\{\delta e, \delta p^i, \delta n\}$ and $\{\delta e, \delta p^i, \delta n, \delta j^i\}$ have the same slow modes. Namely, δj^i contains the irrelevant part for the slow dynamics of relativistic fluids. Moreover, we will illustrate that, even for the Eckart equation, the slow dynamics is actually described by the Landau's variables, $\{\delta e, \delta p^i, \delta n\}$.

Our study is first attempt for the derivation based on the projection operator method. The earlier studies [7–10] assume that the relativistic Boltzmann equation as an underlying microscopic theory. In contrast, we stress that our derivation is independent of microscopic details.

This paper is organized as follows. In Sec. II, we briefly review Mori's projection operator method for readers being unfamiliar with it. Also, we explain conserved charges as the slow variables and discuss those for relativistic hydrodynamics. In Sec. III, we first determine equal-time correlations of the slow variables to determine the property of the equilibrium state. We note that we here consider the fluctuations from *the equilibrium state*. Our determination is based on thermodynamics and Lorentz symmetry on microscopic scale. Then, it is independent of the microscopic detail. In Sec. IV, we derive slow dynamics by the projection operator method for the sets, Eqs. (23) and (24). In Sec. V, we discuss the results obtained in Sec. IV. In particular, we study slow modes of the equations for $\{\delta e, \delta p^i, \delta n\}$ and $\{\delta e, \delta p^i, \delta n, \delta j^i\}$. We explicitly show that these equations have the same modes. Furthermore, we consider the Onsager reciprocal relation in the Eckart equation to illustrate that the slow part of δj coming from $\{\delta e, \delta p^i, \delta n\}$.

II. MORI'S PROJECTION OPERATOR METHOD

In this section, we provide Mori's projection operator method [15, 17, 18, 21, 22]. We can formally extract slow dynamics from microscopic Hamiltonian dynamics by this method. On the microscopic scale, an operator at time t , $\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}$, evolves by the Heisenberg equation,

$$\partial_0 \mathcal{O}(t) = i[H, \mathcal{O}(t)] \equiv i\mathcal{L}\mathcal{O}(t), \quad (25)$$

where \mathcal{L} is the Liouville operator. In the following, we decompose this time evolution equation into slow and fast ones.

A. Projection operator

First, we introduce basic ingredients for the projection operator. Let us consider a many-body system at finite temperature. As an equilibrium distribution, we assume the grand-canonical one. Then, the density matrix is given as

$$\rho_{\text{eq}} \equiv \frac{e^{-\beta(H-\mu N)}}{\text{tr } e^{-\beta(H-\mu N)}}, \quad (26)$$

where H is the Hamiltonian, N is the number operator, μ the chemical potential, and the inverse temperature $\beta = 1/T$. With the density matrix, thermal average of $\mathcal{O}(t)$ is defined as

$$\langle \mathcal{O}(t) \rangle_{\text{eq}} \equiv \text{tr } \rho_{\text{eq}} \mathcal{O}(t) = \langle \mathcal{O}(0) \rangle_{\text{eq}}. \quad (27)$$

Also, we define an inner product of A and B as

$$(A, B) \equiv \frac{1}{\beta} \int_0^\beta d\tau \langle e^{\tau H} A e^{-\tau H} B^\dagger \rangle_{\text{eq}} = \frac{1}{\beta} \int_0^\beta d\tau \langle A(-i\tau) B^\dagger \rangle_{\text{eq}}. \quad (28)$$

Moreover, we introduce a set of slowly-varying operators (slow variables), $\{A_n(t, \mathbf{x})\} = \{A_1, A_2, \dots, A_n\}$. If we can separate the time scale into long- and short-time ones, such operators exist and describe the slow dynamics. Let us consider the slow operators at the initial time $t = 0$, $\{A_n(0, \mathbf{x})\}$. In general, they are not orthogonal to each other. We introduce metric to consider the orthogonal basis:

$$g_{nm}(\mathbf{x} - \mathbf{y}) \equiv (A_n(0, \mathbf{x}), A_m(0, \mathbf{y})). \quad (29)$$

The orthogonal operators, represented with an upper index, is defined as

$$A^n(t, \mathbf{x}) \equiv \int d^3y g^{nm}(\mathbf{x} - \mathbf{y}) A_m(t, \mathbf{y}), \quad (30)$$

where $g^{nm}(\mathbf{x} - \mathbf{y})$ is the inverse of $g_{nm}(\mathbf{x} - \mathbf{y})$. These quantities are orthogonal to those with lower indices,

$$(A_n(0, \mathbf{x}), A^m(0, \mathbf{y})) = \delta_n^m \delta(\mathbf{x} - \mathbf{y}), \quad (31)$$

$$\sum_m \int d^3y g_{nm}(\mathbf{x} - \mathbf{y}) g^{ml}(\mathbf{y} - \mathbf{z}) = \delta_n^l \delta(\mathbf{x} - \mathbf{y}). \quad (32)$$

It is useful to see the relation between the metric and the effective potential $V(\{A_n\})$, which is given as

$$V(\{A_n\}) = \frac{1}{2} \sum_{n,m} \int d\mathbf{x} d\mathbf{y} A_n(\mathbf{x}) g^{nm}(\mathbf{x} - \mathbf{y}) A_m(\mathbf{y}) + \dots, \quad (33)$$

where ... are higher order terms of A_n and can be neglected in the linear regime. We note that we are interested in fluctuations as A_n .

We have prepared the basic ingredients. Let us introduce the projection operator acting on any operators $B(t, \mathbf{x})$ as

$$\mathcal{P}B(t, \mathbf{x}) \equiv \sum_n \int d^3y A_n(0, \mathbf{y}) (B(t, \mathbf{y}), A^n(0, \mathbf{x})). \quad (34)$$

The projection operator extracts the slowly varying part of B , which is determined only by the slow variables $\{A_n\}$. We also define the orthogonal projector as $\mathcal{Q} \equiv 1 - \mathcal{P}$ for later use.

B. Generalized Langevin equation

In this subsection, we derive so-called the generalized Langevin equation. This equation is given by decomposing the Heisenberg equation into slow and fast parts. For the decomposition, we use the following operator identity:

$$\partial_0 e^{i\mathcal{L}t} = e^{i\mathcal{L}t} \mathcal{P}i\mathcal{L} + \int_0^t ds e^{i\mathcal{L}(t-s)} \mathcal{P}i\mathcal{L} e^{\mathcal{Q}i\mathcal{L}s} \mathcal{Q}i\mathcal{L} + e^{\mathcal{Q}i\mathcal{L}t} \mathcal{Q}i\mathcal{L}, \quad (35)$$

which is valid for arbitrary \mathcal{L} and \mathcal{P} [16].

Let us derive this identity. First, we consider the following decomposition:

$$\partial_0 e^{i\mathcal{L}t} = e^{i\mathcal{L}t} i\mathcal{L} = e^{i\mathcal{L}t} \mathcal{P}i\mathcal{L} + e^{i\mathcal{L}t} \mathcal{Q}i\mathcal{L}. \quad (36)$$

Next, we consider the Laplace transform of $\exp(i\mathcal{L}t)$,

$$\int_0^\infty dt e^{-zt} e^{i\mathcal{L}t} = \frac{1}{z - i\mathcal{L}}. \quad (37)$$

Then, we decompose Eq. (37) into

$$\begin{aligned} \frac{1}{z - i\mathcal{L}} &= \frac{1}{z - i\mathcal{L}} (z - \mathcal{Q}i\mathcal{L}) \frac{1}{z - \mathcal{Q}i\mathcal{L}} \\ &= \frac{1}{z - i\mathcal{L}} (z - i\mathcal{L} + \mathcal{P}i\mathcal{L}) \frac{1}{z - \mathcal{Q}i\mathcal{L}} \\ &= \frac{1}{z - \mathcal{Q}i\mathcal{L}} + \frac{1}{z - i\mathcal{L}} \mathcal{P}i\mathcal{L} \frac{1}{z - \mathcal{Q}i\mathcal{L}}. \end{aligned} \quad (38)$$

Performing the inverse Laplace transform, we find the identity,

$$e^{i\mathcal{L}t} = e^{\mathcal{Q}i\mathcal{L}t} + \int_0^t ds e^{i\mathcal{L}(t-s)} P i\mathcal{L} e^{\mathcal{Q}i\mathcal{L}s}. \quad (39)$$

Substituting Eq. (39) into the second term of Eq. (36), we obtain the operator identity, Eq. (35).

Multiplying Eq. (35) by an initial value of the slow operator, we obtain the decomposed equation of motion for $A_n(t) = e^{i\mathcal{L}t} A_n(0)$:

$$\partial_0 A_n(t, \mathbf{x}) = \int d^3y i\Omega_n^m(\mathbf{x} - \mathbf{y}) A_m(t, \mathbf{y}) - \int_0^t ds d^3y \Phi_n^m(t-s, \mathbf{x} - \mathbf{y}) A_m(s, \mathbf{y}) + R_n(t, \mathbf{x}), \quad (40)$$

without any approximations. Here, we introduced the following functions and operator:

$$i\Omega_n^m(\mathbf{x} - \mathbf{y}) \equiv (i\mathcal{L}A_n(0, \mathbf{x}), A^m(0, \mathbf{y})) = -\frac{1}{\beta} i \langle [A_n(0, \mathbf{x}), A^{m\dagger}(0, \mathbf{y})] \rangle_{\text{eq}}, \quad (41)$$

$$\Phi_n^m(t-s, \mathbf{x} - \mathbf{y}) \equiv -\theta(t-s) (i\mathcal{L}R_n(t, \mathbf{x}), A^m(s, \mathbf{y})), \quad (42)$$

$$R_n(t, \mathbf{x}) \equiv e^{it\mathcal{Q}\mathcal{L}} Q i\mathcal{L} A_n(0, \mathbf{x}). \quad (43)$$

Equation (40) is the generalized Langevin equation and has the following properties:

1. Equation (40) is the *operator identity*.
2. The first and second terms in the right-hand side represent the slow motions.
3. The first term corresponds to a time-reversible change.
4. The second term corresponds to a time-irreversible change. Also, this term depends on a past time value, $A_m(s)$, for $s < t$. $\Phi_n^m(t-s, \mathbf{x} - \mathbf{y})$ is called the memory function.
5. The last term is the noise term corresponding to the fast motion. For hydrodynamics, this term is usually neglected. On the other hand, for the Langevin dynamics, we treat this term as a random noise.

It is useful to rewrite Eq. (40) to the equation in momentum space:

$$\partial_0 A_n(t, \mathbf{k}) = i\Omega_n^m(\mathbf{k}) A_m(t, \mathbf{k}) - \int_0^t ds \Phi_n^m(t-s, \mathbf{k}) A_m(s, \mathbf{k}) + R_n(t, \mathbf{k}). \quad (44)$$

For time component, we perform the Laplace transform:

$$A_n(z, \mathbf{k}) = \int dt \int d^3x e^{-zt} e^{-i\mathbf{k}\cdot\mathbf{x}} A_n(t, \mathbf{x}). \quad (45)$$

Then, Eq. (44) becomes

$$zA(z, \mathbf{k}) = i\Omega(\mathbf{k})A(z, \mathbf{k}) - \Phi(z, \mathbf{k})A(z, \mathbf{k}) + R(z, \mathbf{k}) + A(t=0, \mathbf{k}) \quad (46)$$

in the Laplace-momentum space. Here, $A(t=0, \mathbf{k})$ is the initial value and we used matrix notation.

C. Conserved charges as slow variables

Here, we provide why conserved charges are slow and discuss the dynamic variables for the Landau and Eckart frames. The key is that conserved charge densities generally satisfy conservation laws:

$$\partial_0 j^0 = -\partial_i j^i, \quad (47)$$

where j^0 is a conserved charged density, and j^i is its current. In the momentum space, the conservation law becomes

$$\partial_0 j^0 = ik_i j^i. \quad (48)$$

We note that the time change rate of j^0 is proportional to the wavenumber, so that the low-wavenumber components turn out to be slow. Therefore, the change of the conserved charge densities are necessarily slow in the low-wavenumber region, i.e., on macroscopic scale.

Now, let us consider the case of relativistic hydrodynamics. We here have the three conservation laws in Eqs. (1) and (2). From those, we obtain the three conserved charges, the particle number, the energy and the momentum:

$$\partial_0 j^0 = ik_i j^i, \quad (49)$$

$$\partial_0 T^{00} = ik_i T^{0i}, \quad (50)$$

$$\partial_0 T^{i0} = ik_j T^{ij}. \quad (51)$$

We note that the above quantities are the slow variables for the Landau equation. The important point is that the particle current is not conserved:

$$\partial_0 j^i \neq ik_j \Pi^{ij}. \quad (52)$$

Thus, the time change rate is not proportional to the wavenumber. Namely, the particle current is not essentially slow although it is proportional to the fluid velocity for the Eckart frame.

Nevertheless, we note that the particle current has a slow part coming from the conserved charges, because j^i is not orthogonal to $\{j^0, T^{00}, T^{0i}\}$. In other word, the projection of j^i on those does not vanish,

$$\mathcal{P}j^i \neq 0, \quad (53)$$

and gives the slow part. From this slow part, we can derive the linearized Eckart equation as we will show in Sec. IV. Here, we stress that the slow dynamics is essentially determined by the conserved charges, $\{j^0, T^{00}, T^{0i}\}$, even for the Eckart equation.

III. METRIC AND THERMODYNAMIC QUANTITIES

In this section, we discuss relations between the metric g_{nm} and thermodynamic quantities [16, 17, 23]. As discussed in Sec. I, we employ the fluctuations of conserved charges as slow variables, i.e., $A_n = \{\delta e, \delta p^i, \delta n, \delta j^i\}$ with $\delta e \equiv T^{00} - e_0$, $\delta p^i \equiv T^{0i}$, $\delta n \equiv j^0 - n_0$, and $\delta j^i \equiv j^i$ with $e_0 \equiv \langle T^{00} \rangle_{\text{eq}}$ and $n_0 \equiv \langle n \rangle_{\text{eq}}$. We assume that the density matrix at thermal equilibrium is invariant under time reversal transformation, i.e., $\mathcal{T}\rho_{\text{eq}}\mathcal{T}^{-1} = \rho_{\text{eq}}$, where \mathcal{T} is the time reversal operator. The slow variables transform under \mathcal{T} as

$$\mathcal{T}\delta e(t, \mathbf{x})\mathcal{T}^{-1} = \delta e(-t, \mathbf{x}), \quad \mathcal{T}\delta n(t, \mathbf{x})\mathcal{T}^{-1} = \delta n(-t, \mathbf{x}), \quad (54)$$

$$\mathcal{T}\delta p^i(t, \mathbf{x})\mathcal{T}^{-1} = -\delta p^i(-t, \mathbf{x}), \quad \mathcal{T}\delta j^i(t, \mathbf{x})\mathcal{T}^{-1} = -\delta j^i(-t, \mathbf{x}). \quad (55)$$

δe and δn (δp^i and δj^i) are even (odd) operators, so that δn and δe does not mix δp^i and δj^i , i.e., $g_{ep}(\mathbf{k}) = g_{ej}(\mathbf{k}) = g_{np}(\mathbf{k}) = g_{nj}(\mathbf{k}) = 0$.

Since we are interested in the low energy behavior of slow variables, we apply the derivative expansion. The metric is expanded as a power series of k^i ,

$$g_{nm}(\mathbf{k}) = g_{nm} + g_{nm;i}^{(1)}k^i + g_{nm;ij}^{(2)}k^i k^j + \dots, \quad (56)$$

where we assumed $g_{nm}(\mathbf{k})$ is analytic at $\mathbf{k} = \mathbf{0}$; in other words, there are no long range correlations. The only leading terms, g_{nm} , contribute to the linearized hydrodynamic equations at first order, so that we will not consider contributions from the higher order terms.

A. g_{nn} , g_{ee} and g_{en}

First, we focus on g_{nn} , g_{ee} and g_{en} , which are given as

$$g_{ee} = \int d^3x (\delta e(\mathbf{x}), \delta e(\mathbf{0})), \quad g_{nn} = \int d^3x (\delta n(\mathbf{x}), \delta n(\mathbf{0})), \quad g_{en} = g_{ne} = \int d^3x (\delta e(\mathbf{x}), \delta n(\mathbf{0})). \quad (57)$$

By definition, these metric are related the susceptibilities. In fact, g_{nn} can be expressed using the isothermal compressibility k_T as

$$g_{nn} = \frac{1}{T_0 V} \langle (\delta N)^2 \rangle_{\text{eq}} = n_0^2 T_0 k_T, \quad (58)$$

where $\delta N = N - \langle N \rangle_{\text{eq}}$ and V is the volume. g_{ee} and g_{en} are related to the specific heat at constant volume $c_V = \beta_0^2(g_{ee} - g_{ne}^2/g_{nn}) = \beta_0^2/g^{ee}$. The inverse matrices for e and n can be written by the entropy density as a functional of e and n :

$$g^{ee} = -\frac{\partial^2 s}{\partial e^2}, \quad g^{nn} = -\frac{\partial^2 s}{\partial n^2}, \quad g^{ne} = g^{en} = -\frac{\partial^2 s}{\partial e \partial n}, \quad (59)$$

because the effective potential as a functional of δn and δe is given as the entropy density $s(\delta e, \delta n)$, for grand canonical ensembles [16]. Then, the slow variables with upper index are written as

$$A^e = -\left(\frac{\partial^2 s}{\partial e^2}\right) \delta e - \left(\frac{\partial^2 s}{\partial n \partial e}\right) \delta n, \quad (60)$$

$$A^n = -\left(\frac{\partial^2 s}{\partial n \partial e}\right) \delta e - \left(\frac{\partial^2 s}{\partial n^2}\right) \delta n. \quad (61)$$

If we notice the thermodynamic relations

$$\left(\frac{\partial s}{\partial e}\right) = \beta, \quad \left(\frac{\partial s}{\partial n}\right) = -\beta\mu, \quad (62)$$

we see that

$$A^e = -\left(\frac{\partial \beta}{\partial e}\right) \delta e - \left(\frac{\partial \beta}{\partial n}\right) \delta n = -\delta\beta, \quad (63)$$

$$A^n = \left(\frac{\partial(\beta\mu)}{\partial e}\right) \delta e + \left(\frac{\partial(\beta\mu)}{\partial n}\right) \delta n = \delta(\beta\mu). \quad (64)$$

Namely, $-\delta\beta$ and $\delta(\beta\mu)$ are orthonormal to δe and δn :

$$(\delta e(t, \mathbf{r}), \delta(\beta\mu)(t, \mathbf{r}')) = (\delta n(t, \mathbf{r}), \delta\beta(t, \mathbf{r}')) = 0, \quad (65)$$

$$-(\delta e(t, \mathbf{r}), \delta\beta(t, \mathbf{r}')) = (\delta n(t, \mathbf{r}), \delta(\beta\mu)(t, \mathbf{r}')) = \delta(\mathbf{r} - \mathbf{r}'). \quad (66)$$

B. $g_{p^i p^j}$, $g_{p^i j^j}$, and $g_{j^i j^j}$

Next, we consider $g_{p^i p^j}$, $g_{p^i j^j}$ and $g_{j^i j^j}$. We show that two of them, $g_{p^i p^j}$ and $g_{p^i j^j}$, are expressed as the enthalpy and the number density:

$$g_{p^i p^j} = \int d^3x (T^{0i}(0, \mathbf{x}), T^{0j}(0, \mathbf{0})) = \delta^{ij} T_0 h_0, \quad (67)$$

$$g_{p^i j^j} = \int d^3x (T^{0i}(0, \mathbf{x}), j^i(0, \mathbf{0})) = \delta^{ij} T_0 n_0. \quad (68)$$

The same relations are obtained in Ref. [23]. These relations are derived from Lorentz symmetry underlying theory. For an arbitrary Hermitian operator \mathcal{O} , the following identity is satisfied:

$$\int d^3x (T^{0i}(0, \mathbf{x}), \mathcal{O}) = (P^i, \mathcal{O}) = i([H, K^i], \mathcal{O}) = -iT\langle [K^i, \mathcal{O}] \rangle_{\text{eq}}, \quad (69)$$

where K^i is the boost operator, $[H, K^i] = -iP^i$ and the following Kubo's identity was employed:

$$([H, A], B) = -T\langle [A, B^\dagger] \rangle_{\text{eq}}. \quad (70)$$

Since $T^{\mu\nu}(x)$ and $j^\mu(x)$ are Lorentz tensor and vector, respectively, these transform under Lorentz transformation as

$$[L^{\mu\nu}, T^{\lambda\rho}(x)] = i(x^\mu \partial^\nu - x^\nu \partial^\mu) T^{\lambda\rho}(x) - i(\eta^{\mu\lambda} T^{\nu\rho}(x) - \eta^{\nu\lambda} T^{\mu\rho}(x) + \eta^{\mu\rho} T^{\lambda\nu} - \eta^{\nu\rho} T^{\lambda\mu}(x)), \quad (71)$$

$$[L^{\mu\nu}, j^\rho(x)] = i(x^\mu \partial^\nu - x^\nu \partial^\mu) j^\rho(x) - i(\eta^{\mu\rho} j^\nu(x) - \eta^{\nu\rho} j^\mu(x)), \quad (72)$$

where $L^{\mu\nu}$ is the charge of Lorentz symmetry, and $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the (inverse) Minkowski metric. For the Lorentz boost $K^i = L^{i0}$, they obey

$$[K^i, T^{0j}(x)] = -i(x^0 \partial^i - x^i \partial^0)T^{0j}(x) + iT^{ij} - i\eta^{ij}T^{00}(x), \quad (73)$$

$$[K^i, j^j(x)] = -i(x^0 \partial^i - x^i \partial^0)j^j(x) - i\eta^{ij}n(x). \quad (74)$$

Therefore, the thermal averages for these commutators satisfy

$$\langle [K^i, T^{0j}(x)] \rangle_{\text{eq}} = i\langle T^{ij}(x) - \eta^{ij}T^{00}(x) \rangle_{\text{eq}} = i\delta^{ij}h_0, \quad (75)$$

$$\langle [K^i, j^{0j}(x)] \rangle_{\text{eq}} = -i\langle \eta^{ij}n(x) \rangle_{\text{eq}} = i\delta^{ij}n_0. \quad (76)$$

Inserting Eqs. (75) and (76) into Eq. (69), we arrive at Eqs. (67) and (68). These identities enable us to relate two-point functions to one-point functions.

IV. APPLICATION OF MORI'S PROJECTION OPERATOR METHOD TO RELATIVISTIC HYDRODYNAMICS

In this section, we apply Mori's projection operator method to relativistic hydrodynamic systems, and derive equations of motion for $\{\delta e, \delta p^i, \delta n\}$ and $\{\delta e, \delta p^i, \delta n, \delta j^i\}$. We first show that the set, $\{\delta e, \delta p^i, \delta n\}$, gives the linearized Landau equation. For the Eckart equations, we introduce the current of the conserved charge, δj^i , which is proportional to the fluid velocity in the Eckart frame, in addition to δp^i and δn . We employ the derivative expansion and keep the spatial and time derivative to the second order, i.e., $\partial_0, \nabla, \nabla^2, \partial_0 \nabla$ and ∂_0^2 . We will drop the noise term $R_n(t, \mathbf{x})$ in the equation of motion. This term is irrelevant in the time evolution of the expectation value. If one is interested in stochastic hydrodynamics, one may keep the noise term [24].

A. Linearized Landau equation

First, we derive the linearized Landau equation. For this purpose, we choose δe , δp and δn as slow variables. Since δp^i is chosen as a slow variable, the equation for $\partial_0 \delta e$ does not contain dissipative terms,

$$\partial_0 \delta e = -\nabla \cdot \delta \mathbf{p} = -h_0 \nabla \cdot \delta \mathbf{v}_L, \quad (77)$$

where we defined the fluid velocity $\delta \mathbf{v}_L \equiv \delta \mathbf{p}/h_0$. This equation is nothing but energy conservation law, Eq. (12). This can be confirmed by the the following calculation:

$$\begin{aligned} i\Omega_e^p(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} (i\mathcal{L}\delta e(\mathbf{x}), \delta p^j(\mathbf{0})) g^{p^j p^i}(\mathbf{k}) \\ &= \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} (-\nabla^j \delta p^j(\mathbf{x}), \delta p^i(\mathbf{0})) g^{p^j p^i}(\mathbf{k}) \\ &= -ik^l g_{p^l p^j}(\mathbf{k}) g^{p^j p^i}(\mathbf{k}) \\ &= -ik^i. \end{aligned} \quad (78)$$

Therefore, the reversible term becomes $-\nabla \cdot \delta \mathbf{p}$. The memory function vanishes because $i\mathcal{L}\delta e$ turns out to be $-i\mathbf{k} \cdot \delta \mathbf{p}$, and then $\mathcal{Q}i\mathcal{L}\delta e = 0$ [see Eqs. (42) and (43)].

Let us move onto the equation for $\partial_0 \delta n$. For the reversible part, the only $i\Omega_{np^i}$ survives in Ω from time reversal symmetry, which is

$$\begin{aligned} i\Omega_{np^i}(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} (i\mathcal{L}\delta n(\mathbf{x}), \delta p^i(\mathbf{0})) \\ &= \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} (-\nabla^j j^j(\mathbf{x}), \delta p^i(\mathbf{0})) \\ &= -ik^j g_{j p^i} + \mathcal{O}(k^3) \\ &= -ik^i T_0 n_0 + \mathcal{O}(k^3). \end{aligned} \quad (79)$$

For the memory function, we keep it up to of order k^2 . Φ_{ne} vanishes as the same as previous due to $\mathcal{Q}i\mathcal{L}\delta e = 0$. Φ_{np} is of order k^3 from tensor structure, which can be neglected. Therefore, we may only consider Φ_{nn} . Since we are interested in slow dynamics, we also expand the memory function in terms of z in addition to k :

$$\begin{aligned}\Phi_{nn}(z, \mathbf{k}) &= \int_0^\infty dt \int d^3x e^{-zt} e^{-i\mathbf{k}\cdot\mathbf{x}} (e^{it\mathcal{Q}\mathcal{L}} \mathcal{Q}i\mathcal{L}\delta n(0, \mathbf{x}), i\mathcal{L}\delta n(0, \mathbf{0})) \\ &= k^i k^j \int_0^\infty dt \int d^3x e^{-zt} e^{-i\mathbf{k}\cdot\mathbf{x}} (e^{it\mathcal{Q}\mathcal{L}} \mathcal{Q}\delta j^i(0, \mathbf{x}), \delta j^j(0, \mathbf{0})) \\ &\simeq \mathbf{k}^2 \int_0^\infty dt \int d^3x (\delta j^i(t, \mathbf{x}) - \frac{n_0}{h_0} \delta p^i(t, \mathbf{x}), \delta j^i(0, \mathbf{0}) - \frac{n_0}{h_0} \delta p^i(0, \mathbf{0})) \\ &= \mathbf{k}^2 \lambda \left(\frac{n_0 T_0}{h_0} \right)^2 \equiv \mathbf{k}^2 \tilde{\lambda},\end{aligned}\tag{80}$$

where \simeq denotes the approximation of order k^2 and z^0 . The approximation of order z^0 corresponds to the Markov approximation, i.e., in the coordinate space, $\Phi_{nn}(t, \mathbf{x}) \simeq -\tilde{\lambda} \nabla^2 \delta(t) \delta^{(3)}(\mathbf{x})$. We defined the thermal conductivity λ as

$$\lambda \equiv \left(\frac{h_0}{n_0 T_0} \right)^2 \int_0^\infty dt \int d^3x (\delta j^i(t, \mathbf{x}) - \frac{n_0}{h_0} \delta p^i(t, \mathbf{x}), \delta j^i(0, \mathbf{0}) - \frac{n_0}{h_0} \delta p^i(0, \mathbf{0})),\tag{81}$$

and we used

$$\mathcal{Q}\delta j^i(t, \mathbf{x}) \simeq \delta j^i(t, \mathbf{x}) - \frac{n_0}{h_0} \delta p^i(t, \mathbf{x})\tag{82}$$

in the leading order of the derivative expansion. We note that the second term is important to remove the contribution of zero mode from δj^i . As a result, we arrive at the equation for $\partial_0 \delta n$ as

$$\partial_0 \delta n = -n_0 \nabla \cdot \delta \mathbf{v}_L + \tilde{\lambda} \nabla^2 \delta(\beta \mu).\tag{83}$$

This equation coincides with Eq. (11).

Similarly, for δp^i , the reversible term $i\Omega_{pe} = -ik^i T_0 h_0$, $i\Omega_{pn} = -ik^i T_0 n_0$, so that

$$-i\Omega_{pe} F_e - i\Omega_{pn} F_n = -ik^i (-T_0 h_0 \delta \beta + T_0 n_0 \delta(\beta \mu)) = -ik^i \delta P,\tag{84}$$

where we used Gibbs-Duhem's relation,

$$\delta P = \frac{h_0}{T_0} \delta T + T_0 n_0 \delta(\beta \mu).\tag{85}$$

For the dissipative terms, Φ_{pe} vanishes, and $\Phi_{pn} \sim k^3$ from tensor structure can be neglected as the same as the previous. Therefore, only Φ_{pp} survives in the leading order, which is evaluated as

$$\begin{aligned}\Phi_{p^i p^k}(z, \mathbf{k}) &\simeq k^j k^l \int dt \int d^3x (T^{ij}(t, \mathbf{x}), T^{kl}(0, \mathbf{x})) \\ &= T_0 \left(\zeta + \frac{1}{3} \eta \right) k^i k^k + T_0 \eta \mathbf{k}^2 \delta^{ik},\end{aligned}\tag{86}$$

where we used the same approximation in Eq. (80). The shear and bulk viscosities are defined by Kubo formula as

$$\eta = \beta_0 \int_0^\infty dt \int d^3x (T^{12}(t, \mathbf{x}), T^{12}(0, \mathbf{0})),\tag{87}$$

$$\zeta - \frac{2}{3} \eta = \beta_0 \int_0^\infty dt \int d^3x (T^{11}(t, \mathbf{x}), T^{22}(0, \mathbf{0})).\tag{88}$$

Noting that $A^{p^i} = -\beta_0 \delta v_L^i = -(\beta_0/h_0) \delta p^i$, we obtain

$$\partial_0 \delta \mathbf{p} = -\nabla \delta P + \eta \nabla^2 \delta \mathbf{v}_L + \left(\zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \delta \mathbf{v}_L),\tag{89}$$

which coincides with Eq. (13). We have shown that the linearized Landau equations, Eqs (77), (83) and (89), are derived by choosing δe , δn , and δp^i as slow variables.

Before closing this subsection, let us consider the detail of $\Phi_{nn}(z, \mathbf{k})$. The memory function can be written as

$$\Phi(z, \mathbf{k}) = -(\ddot{\Xi}(z, \mathbf{k}) - i\Omega(\mathbf{k})\dot{\Xi}(z, \mathbf{k})) \frac{1}{1 + \dot{\Xi}(z, \mathbf{k})}, \quad (90)$$

where we defined

$$\Xi(t) \equiv (A_n(t), A^m), \quad (91)$$

$$\dot{\Xi}(t) \equiv (i\mathcal{L}A_n(t), A^m), \quad (92)$$

$$\ddot{\Xi}(t) \equiv ((i\mathcal{L})^2 A_n(t), A^m). \quad (93)$$

For the derivation of Eq. (90), see Appendix B. Since the time derivative of a conserved charge variable is slow, $\dot{\Xi}$ is of order k ; then, we can estimate $1/(1 + \dot{\Xi}) = 1 + \mathcal{O}(k)$. And then, $\Phi_{nn}(z, \mathbf{k})$ becomes

$$\begin{aligned} \Phi_{nn}(z, \mathbf{k}) &\simeq - \left[\ddot{\Xi}(z, \mathbf{k}) - i\Omega(\mathbf{k})\dot{\Xi}(z, \mathbf{k}) \right]_{nn} \\ &\simeq \mathbf{k}^2 \left(\Xi_{jj}(z, \mathbf{0}) - \frac{n_0}{h_0} \Xi_{pj}(z, \mathbf{0}) \right). \end{aligned} \quad (94)$$

From $\dot{\Xi} = -1 + z\Xi$,

$$\Xi_{pj}(z, \mathbf{0}) = -\frac{1}{z}g_{pj} = -\frac{1}{z}n_0T_0, \quad (95)$$

where we used $\dot{\Xi}_{pj}(z, \mathbf{0}) = 0$. Therefore, we can write

$$\Phi_{nn}(z, \mathbf{k}) \simeq \mathbf{k}^2 \left(\Xi_{jj}(z, \mathbf{0}) - \frac{n_0^2 T_0}{zh_0} \right) \equiv \mathbf{k}^2 \tilde{\lambda}(z), \quad (96)$$

where we defined the frequency-dependent-thermal conductivity $\tilde{\lambda}(z)$. At $z = 0$, $\tilde{\lambda}(0)$ coincide with $\tilde{\lambda}$. This expression will be used in the next subsection to derive the linearized Eckart equation.

B. Linearized Eckart equation

Next, we derive the linearized Eckart equation. The charge does not dissipate in the Eckart equation, which implies that the fluid velocity is chosen as $\delta j^i/n$. Therefore, we choose δj^i as a slow variable in addition to $\{\delta e, \delta p^i, \delta n\}$. In order to derive the Eckart equation, we first derive the equations of motion for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$; and then, we remove the degrees of freedom of δp^i .

The equation of time evolution for δe and δn are trivial thanks to the conservation laws,

$$\partial_0 \delta e = -\nabla \cdot \delta \mathbf{p}, \quad (97)$$

$$\partial_0 \delta n = -\nabla \cdot \delta \mathbf{j}. \quad (98)$$

As usual, we calculate the reversible terms:

$$i\Omega_{ep^i}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} (i\mathcal{L}\delta e(\mathbf{x}), \delta p^i(\mathbf{0})) = -ik^i T_0 h_0 + \mathcal{O}(k^3), \quad (99)$$

$$i\Omega_{ej^i}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} (i\mathcal{L}\delta n(\mathbf{x}), \delta j^i(\mathbf{0})) = -ik^i T_0 n_0 + \mathcal{O}(k^3), \quad (100)$$

$$i\Omega_{nj^i}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} (i\mathcal{L}\delta n(\mathbf{x}), \delta j^i(\mathbf{0})) = -ik^j g_{jj^i} + \mathcal{O}(k^3). \quad (101)$$

Here the explicit form of g_{jj^i} is not obtained; however, it is irrelevant in the leading order of the fluid equations, as will be seen later. The reversible term for $\partial_0 \delta p^i$ is the same as that of the Landau equation in Eq. (84). The reversible term for $\partial_0 \delta j^i$ becomes

$$i\Omega_{jn}A^n + i\Omega_{je}A^e = -ik^i \left(g_{jj} \delta(\beta\mu) - T_0 n_0 \delta\beta \right). \quad (102)$$

Since δj^i and δp^i are chosen as slow variables, the inverse metric contain mixing terms:

$$\begin{pmatrix} g^{pp} & g^{pj} \\ g^{jp} & g^{jj} \end{pmatrix} = \frac{1}{g_{pp}g_{jj} - g_{pj}g_{jp}} \begin{pmatrix} g_{jj} & -g_{pj} \\ -g_{jp} & g_{pp} \end{pmatrix} = \frac{\beta_0}{h_0\beta_0g_{jj} - n_0^2} \begin{pmatrix} \beta_0g_{jj} & -n_0 \\ -n_0 & h_0 \end{pmatrix}. \quad (103)$$

Then, the conjugate variables for δj^i and δp^i are

$$A^j = \frac{\beta_0}{h_0\beta_0g_{jj} - n_0^2} (-n_0\delta p^j + h_0\delta j^j) = -n_0g^{jj}(\delta v_L^j - \delta v_E^j), \quad (104)$$

$$A^p = \frac{\beta_0}{h_0\beta_0g_{jj} - n_0^2} (\beta_0g_{jj}\delta p^p - n_0\delta j^p) = \beta_0\delta v_E^p + \beta_0g_{jj}h_0A^j = \beta_0\delta v_L^p - \frac{n_0}{h_0}A^j, \quad (105)$$

where we defined the fluid velocity in the Eckart frame as $\delta v_E^i \equiv \delta j^i/n_0$. The memory functions appearing in $\partial_0\delta p^i$ are Φ_{pp} and Φ_{pj} , which are both of order k^2 . Here, we assume that A^j is of order k , which will be checked later. Then, $\Phi_{pj}A^j$ can be neglected because it is of order k^3 . Similarly, $\Phi_{pp}A^p \simeq -\beta\Phi_{pp}\delta\mathbf{v}_L$. Therefore,

$$\partial_0\delta\mathbf{p} = -\nabla\delta P + \eta\nabla^2\delta\mathbf{v}_L + \left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\delta\mathbf{v}_L). \quad (106)$$

This is the same equation as the Landau one.

Finally, let us consider the equation for $\partial_0\delta j^i$, which is written in the Laplace space:

$$\begin{aligned} zn_0\delta v_E^i &= i\Omega_{jn}A^n + i\Omega_{je}A^e - \Phi_{jj}A^j - \Phi_{jp}A^p + n_0\delta v_E^i(t=0) \\ &= -g_{jj}\nabla^i\delta(\beta\mu) - \frac{n_0}{T_0}\nabla^i\delta T - \Phi_{jj}A^j - \Phi_{jp}A^p + n_0\delta v_E^i(t=0), \end{aligned} \quad (107)$$

where $\delta\mathbf{v}_E(t=0)$ is the initial value of the fluctuation. The important point is that Φ_{jj} is not slow and gives a contribution of order k^0 . This equation will give the relation between these fluid velocities. Since we are interested in the first order hydrodynamic equation, it is enough to take into account the difference of the fluid velocity up to of order k^1 . In this order, we can neglect Φ_{jp} because it is of order k^2 .

Let us, now, estimate Φ_{jj} . From Eq. (90), we obtain

$$\begin{aligned} \Phi_{jj}(z, \mathbf{k}) &= \left[-(\ddot{\Xi}(z, \mathbf{k}) - i\Omega(\mathbf{k})\dot{\Xi}(z, \mathbf{k})) \frac{1}{1 + \dot{\Xi}(z, \mathbf{k})} \right]_{jj} \\ &= \left[-z + \frac{1}{\Xi(z, \mathbf{k})} + i\Omega(\mathbf{k}) \right]_{jj} \\ &= -zg_{jj} + \left[\frac{1}{\Xi(z, \mathbf{k})} \right]_{jj}, \end{aligned} \quad (108)$$

where we used $\ddot{\Xi} = z\dot{\Xi} - i\Omega$, $\dot{\Xi} = z\Xi - 1$, and $i\Omega_{jj}(\mathbf{k}) = 0$. We may estimate $\Phi_{jj}(z, \mathbf{k})$ at $\mathbf{k} = \mathbf{0}$ in the leading order. First, we consider $\Xi_{ep^i}(z, \mathbf{k})$. The energy conservation provides $z\Xi_{ep^i}(z, \mathbf{k}) = -ik^j\Xi_{pj^i}(z, \mathbf{k})$. Thus, at $\mathbf{k} = \mathbf{0}$, $\Xi_{ep^i}(z, \mathbf{0}) = 0$. Similarly, one can show $\Xi_{ej}(z, \mathbf{0}) = \Xi_{np}(z, \mathbf{0}) = \Xi_{nj}(z, \mathbf{0}) = 0$. Therefore, we may consider the terms with δp^i or δj^i . They are estimated at $\mathbf{k} = \mathbf{0}$ as

$$\begin{pmatrix} \Xi_{pp}(z, \mathbf{0}) & \Xi_{pj}(z, \mathbf{0}) \\ \Xi_{jp}(z, \mathbf{0}) & \Xi_{jj}(z, \mathbf{0}) \end{pmatrix} = \frac{1}{z} \begin{pmatrix} g_{pp} & g_{pj} \\ g_{jp} & zg_{jj}(z, \mathbf{0}) \end{pmatrix} = \frac{1}{z} \begin{pmatrix} h_0T_0 & n_0T_0 \\ n_0T_0 & \frac{n_0^2T_0}{h_0} + z\tilde{\lambda}(z) \end{pmatrix}, \quad (109)$$

where we used $\dot{\Xi}(z, \mathbf{0}) = -1 + z\Xi(z, \mathbf{0}) = 0$, and Eq. (96). Taking into account the metric, we find

$$\Phi_{jj}(z, \mathbf{0}) = \frac{(g_{jj}h_0 - n_0^2T_0)^2}{h_0^2\tilde{\lambda}(z)} - \frac{1}{h_0}(hg_{jj} - n_0^2T_0)z = \frac{1}{(g^{jj})^2\tilde{\lambda}(z)} - \frac{1}{g^{jj}}z. \quad (110)$$

Then, the equation of motion becomes

$$n_0z\delta\mathbf{v}_E = -g_{jj}\nabla\delta(\beta\mu) - \frac{n_0}{T_0}\nabla\delta T + \frac{n_0}{g^{jj}\tilde{\lambda}(z)}(\delta\mathbf{v}_L - \delta\mathbf{v}_E) - n_0z(\delta\mathbf{v}_L - \delta\mathbf{v}_E) + n_0\delta\mathbf{v}_E(t=0). \quad (111)$$

From Eq. (111), we obtain

$$\begin{aligned} h_0 \delta \mathbf{v}_L &= h_0 \delta \mathbf{v}_E + \frac{h_0}{n_0} g^{jj} \tilde{\lambda}(z) \left(n_0 z \delta \mathbf{v}_L + g_{jj} \nabla \delta(\beta \mu) + \frac{n_0}{T_0} \nabla \delta T - n_0 \delta \mathbf{v}_E(t=0) \right) \\ &= h_0 \delta \mathbf{v}_E - \lambda(z) (T_0 z \delta \mathbf{v}_L + \nabla \delta T) + \lambda(z) T_0^2 g^{pp} (h_0 z \delta \mathbf{v}_L + \nabla \delta P) - g^{jj} \frac{n_0^2 T_0^2}{h_0} \lambda(z) \delta \mathbf{v}_E(t=0). \end{aligned} \quad (112)$$

The third term in the last line is estimated as $h_0 z \delta \mathbf{v}_L + \nabla \delta P = h_0 \delta \mathbf{v}_L(t=0) + \mathcal{O}(\nabla^2)$ from the equation of motion, Eq. (106). Then

$$\begin{aligned} h_0 \delta \mathbf{v}_L &= h_0 \delta \mathbf{v}_E - \lambda(z) (T_0 z \delta \mathbf{v}_L + \nabla \delta T) + \lambda(z) \left(T_0^2 g^{pp} h_0 \delta \mathbf{v}_L(t=0) - g^{jj} \frac{n_0^2 T_0^2}{h_0} \delta \mathbf{v}_E(t=0) \right) \\ &= h_0 \delta \mathbf{v}_E - \lambda(z) (T_0 z \delta \mathbf{v}_L + \nabla \delta T - T_0 \delta \mathbf{v}_L(t=0)) - \frac{T_0^2 n_0}{h_0} \lambda(z) A^j(t=0). \end{aligned} \quad (113)$$

This equation is exact for an arbitrary z in the first order of the derivative expansion of spatial coordinate. We note that the equations (97), (98), (106) and (113) are those of motion for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$. These equations are first derived in this paper by the projection operator method. We also note that these include the Landau and Eckart frames in the view point of the projection operator method.

Now, let us derive the Eckart equation. We assume that there is no conjugate variable for j^i at the initial time, i.e., $A^j(t=0) = 0$. We also assume that the change of the variable is so slow that z expansion is applicable:

$$h_0 \delta \mathbf{v}_L = h_0 \delta \mathbf{v}_E - \lambda(T_0 z \delta \mathbf{v}_L + \nabla \delta T - \delta \mathbf{v}_L(t=0)). \quad (114)$$

In the coordinate space, this equation becomes

$$h_0 \delta \mathbf{v}_L = h_0 \delta \mathbf{v}_E - \lambda(T_0 \partial_0 \delta \mathbf{v}_L + \nabla \delta T). \quad (115)$$

From this equation, one can confirm $A^{j^i} = -n_0 g^{jj} (\delta v_L^i - \delta v_E^i) = n_0 g^{jj} (T_0 \partial_0 \delta v_L^i + i k^i \delta T) / h_0 \sim k$. Therefore, the assumption that A^j is of order k is consistent for slow dynamics. Solving Eq. (115) for $h_0 \delta \mathbf{v}_L$, we obtain

$$\begin{aligned} h_0 \delta \mathbf{v}_L &= \frac{1}{1 + \lambda \frac{T_0}{h_0} \partial_0} (h_0 \delta \mathbf{v}_E - \lambda \nabla \delta T) \\ &\simeq h_0 \delta \mathbf{v}_E - \lambda (T_0 \partial_0 \delta \mathbf{v}_E + \nabla \delta T), \end{aligned} \quad (116)$$

where we dropped $\partial_0 \nabla T$ in the last line because it is higher order. Inserting Eq. (112) into Eq. (106), we find

$$\partial_0 (h_0 \delta \mathbf{v}_E - \lambda (T_0 \partial_0 \delta \mathbf{v}_E + \nabla \delta T)) = -\nabla \delta P + \eta \nabla^2 \delta \mathbf{v}_E + \left(\zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \delta \mathbf{v}_E). \quad (117)$$

This equation is equal to the linearized Eckart equation, Eq. (19).

V. DISCUSSION

Here, we discuss the results obtained in the previous section. In the derivation of the Eckart equation, we eliminate the momentum density from the equations for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$. Then, the Eckart equation has $\{\delta e, \delta n, \delta j^i\}$ as the dynamic variables, apparently. Nevertheless, the Landau's variables $\{\delta e, \delta p^i, \delta n\}$ describe the slow dynamics, even for the Eckart equation. To illustrate this fact, we consider the Onsager reciprocal relation in the Eckart equation. If we regard $\{\delta e, \delta n, \delta j^i\}$ as independent variables, the reciprocal relation is violated. In contrast, if we regard δp^i as an independent variable instead of δj^i , we will find that the relation is satisfied. This result shows that the slow part of δj^i is determined by $\{\delta e, \delta p^i, \delta n\}$.

Also, we discuss the instability of the Eckart equation. The Eckart equation has an unstable mode. However, we will find that the equations for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$ do not have the unstable mode and also have the same mode as the Landau equation does. We will show that the unstable mode arises from the expansion about ∂_0 in Eq. (116).

A. Independent variables of the Eckart equation

Let us consider the correlations, $(\partial_0 \delta e, \delta n)$ and $(\partial_0 \delta n, \delta e)$. The important point is that the correlations must satisfy the relation

$$(\partial_0 \delta e, \delta n) = (\partial_0 \delta n, \delta e), \quad (118)$$

which comes from the time reversal properties of δn , δe and the equilibrium state. We note that this relation is equivalent to the Onsager reciprocal relation in the linear regime [25–27]. From the Eckart equations, Eqs. (17)–(19), the correlations are written as

$$(\partial_0 \delta n(t, \mathbf{r}), \delta e(t, \mathbf{r}')) = -n_0 \nabla \cdot (\delta \mathbf{v}_E(t, \mathbf{r}), \delta e(t, \mathbf{r}')), \quad (119)$$

$$\begin{aligned} (\partial_0 \delta e(t, \mathbf{r}), \delta n(t, \mathbf{r}')) &= -h_0 \nabla \cdot (\delta \mathbf{v}_E(t, \mathbf{r}), \delta n(t, \mathbf{r}')) \\ &+ \lambda \left(\nabla^2 (\delta T(t, \mathbf{r}), \delta n(t, \mathbf{r}')) + T_0 \nabla \cdot (\partial_0 \delta \mathbf{v}_E(t, \mathbf{r}), \delta n(t, \mathbf{r}')) \right). \end{aligned} \quad (120)$$

Now, to eliminate the time derivative in the last term of Eq. (120), we use the derivative expansion. Namely, we approximate

$$\lambda T_0 \nabla \cdot \partial_0 \delta \mathbf{v}_E = \lambda T_0 \nabla \cdot h_0^{-1} (-\nabla \delta P + \dots) \quad (121)$$

$$\simeq -\frac{\lambda T_0}{h_0} \nabla^2 \delta P, \quad (122)$$

where ... denotes the second and higher order terms about the derivative, like $\eta \nabla^2 \delta \mathbf{v}$. Here, we first used Eq. (19) for $\partial_0 \delta \mathbf{v}_E$ and next neglect the second-order terms, which finally yields third-order terms. Thus, we obtain

$$(\partial_0 \delta e(t, \mathbf{r}), \delta n(t, \mathbf{r}')) \simeq -h_0 \nabla \cdot (\delta \mathbf{v}_E(t, \mathbf{r}), \delta n(t, \mathbf{r}')) - \lambda \left(\frac{n_0 T_0^2}{h_0} \right) \nabla^2 (\delta(\beta \mu)(t, \mathbf{r}), \delta n(t, \mathbf{r}')), \quad (123)$$

where we also used the Gibbs-Duhem relation, Eq. (85). Let us here estimate the equal-time correlations, $(\delta \mathbf{v}_E, \delta n)$, $(\delta \mathbf{v}_E, \delta e)$ and $(\delta(\beta \mu), \delta n)$. We assume that the time-reversal property of $\delta \mathbf{v}_E$ ($\delta \mathbf{j}$) is odd, according to Eq. (55). Then, $\delta \mathbf{v}_E$ is orthogonal to δn :

$$(\delta \mathbf{v}_E, \delta e) = (\delta \mathbf{v}_E, \delta n) = 0, \quad (124)$$

by the time-reversal symmetry. On the other hand, $(\delta(\beta \mu), \delta n)$ turns out to be $\delta(\mathbf{r} - \mathbf{r}')$ from Eq. (66). Finally, we obtain the correlations,

$$(\partial_0 \delta n(t, \mathbf{r}), \delta e(t, \mathbf{r}')) = 0, \quad (125)$$

$$(\partial_0 \delta e(t, \mathbf{r}), \delta n(t, \mathbf{r}')) = -\lambda \left(\frac{n_0 T_0^2}{h_0} \right) \nabla^2 \delta(\mathbf{r} - \mathbf{r}'). \quad (126)$$

We see that the Onsager relation, Eq. (118), is violated.

Next, let us regard $\delta \mathbf{v}_E$ as a function of $\{\delta e, \delta p^i, \delta n\}$. Namely, we consider $\delta \mathbf{p}$ as an independent variables, instead of $\delta \mathbf{j}$. Then, we now use Eq. (116), which gives the relation between $\delta \mathbf{j}$ and $\delta \mathbf{p}$:

$$\begin{aligned} \delta \mathbf{p} &= h_0 \delta \mathbf{v}_E - \lambda (T \partial_0 \delta \mathbf{v}_L + \nabla \delta T) \\ &\simeq h_0 \delta \mathbf{v}_E - \lambda \left(\frac{n_0 T_0^2}{h_0} \right) \nabla \delta(\beta \mu), \end{aligned} \quad (127)$$

where we used the derivative expansion and the Gibbs-Duhem relation in the second line. Solving the above relation about $\delta \mathbf{v}_E$, we obtain $\delta \mathbf{v}_E$ as the function of $\{\delta n, \delta e, \delta \mathbf{p}\}$:

$$\delta \mathbf{v}_E(\delta n, \delta e, \delta \mathbf{p}) = \frac{1}{h_0} \delta \mathbf{p} + \lambda \left(\frac{n_0 T_0^2}{h_0^2} \right) \nabla \delta(\beta \mu). \quad (128)$$

Here, we note that the time-reversal property of $\delta \mathbf{v}_E(\delta e, \delta p^i, \delta n)$ is not odd because those of $\delta \mathbf{p}$ and $\delta(\beta \mu)$ are odd and even, respectively. Substituting Eq. (128) into Eqs. (119) and (123), we find

$$(\partial_0 \delta n(t, \mathbf{r}), \delta e(t, \mathbf{r}')) = -\frac{n_0}{h_0} \nabla \cdot (\delta \mathbf{p}(t, \mathbf{r}), \delta e(t, \mathbf{r}')) - \lambda \left(\frac{n_0 T_0}{h_0} \right)^2 \nabla^2 (\delta(\beta \mu)(t, \mathbf{r}), \delta e(t, \mathbf{r}')), \quad (129)$$

$$(\partial_0 \delta e(t, \mathbf{r}), \delta n(t, \mathbf{r}')) = (\delta \mathbf{p}(t, \mathbf{r}), \delta n(t, \mathbf{r}')). \quad (130)$$

Here, we note that $\delta(\beta\mu)$ is orthogonal to δe by Eq. (65). Then, if we assume the time reversal property of $\delta\mathbf{p}$ is odd, we see that the Onsager relation is satisfied:

$$(\partial_0 \delta n(t, \mathbf{r}), \delta e(t, \mathbf{r}')) = (\partial_0 \delta e(t, \mathbf{r}), \delta n(t, \mathbf{r}')) = 0. \quad (131)$$

Why cannot we regard $\delta\mathbf{p}$ as a function of $\{\delta n, \delta e, \delta\mathbf{j}\}$? The reason is that $\delta\mathbf{j}$ is not essentially slow, and then the slow motion of that turn out to be described by the actual slow variables. Thus, the time reversal property differs from the original operator at the starting point of Sec. III. Actually, we can show that the projection of $\delta\mathbf{j}$ on $\{\delta n, \delta e, \delta\mathbf{p}\}$ gives Eq. (128). The derivation is given in the Appendix C. Namely, the actual expression of Eq. (128) is given as

$$\mathcal{P}(\delta\mathbf{j}(t)) = \frac{n_0}{h_0} \delta\mathbf{p}(t) - \lambda \left(\frac{n_0 T_0}{h_0} \right)^2 \nabla \left[\left(\frac{\partial(\beta\mu)}{\partial n} \right)_e \delta n(t) + \left(\frac{\partial(\beta\mu)}{\partial e} \right)_n \delta e(t) \right], \quad (132)$$

where \mathcal{P} is the projector on $\{\delta n, \delta e, \delta\mathbf{p}\}$, i.e., $\delta\mathbf{j}$ ($\delta\mathbf{v}_E$) in the Eckart equation is projected one and differs from the original operator. In consequence, the slow variables for the Landau frame actually describe the slow dynamics even for the Eckart frame.

B. Modes of the equations for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$

Now, we study modes of the equations for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$, Eqs. (97), (98), (106) and (113). In the Fourier space, the equations are written as the following matrix form:

$$M \begin{pmatrix} \delta e(\omega, \mathbf{k}) \\ \delta n(\omega, \mathbf{k}) \\ \delta p_{\parallel}(\omega, \mathbf{k}) \\ \delta j_{\parallel}(\omega, \mathbf{k}) \\ \delta \mathbf{p}_{\perp}(\omega, \mathbf{k}) \\ \delta \mathbf{j}_{\perp}(\omega, \mathbf{k}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (133)$$

where we decomposed $\delta\mathbf{p}$ and $\delta\mathbf{j}$ into the longitudinal and transverse components:

$$\delta p_{\parallel} = \hat{\mathbf{k}} \cdot \delta\mathbf{p}, \quad \delta \mathbf{p}_{\perp} = \delta\mathbf{p} - \delta p_{\parallel} \hat{\mathbf{k}}, \quad (134)$$

$$\delta j_{\parallel} = \hat{\mathbf{k}} \cdot \delta\mathbf{j}, \quad \delta \mathbf{j}_{\perp} = \delta\mathbf{j} - \delta j_{\parallel} \hat{\mathbf{k}}. \quad (135)$$

Also, we introduced the matrix

$$M \equiv \begin{pmatrix} -i\omega & 0 & ik & 0 & 0 & 0 \\ 0 & -i\omega & 0 & ik & 0 & 0 \\ ik\alpha_{Pe} & ik\alpha_{Pn} & -i\omega + k^2\Gamma_{\parallel} & 0 & 0 & 0 \\ ikik\alpha_{Te} & ik\alpha_{Tn} & -i\omega + \Gamma_j & -(h_0/n_0)\Gamma_j & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\omega + k^2(\eta/h_0) & 0 \\ 0 & 0 & 0 & 0 & -i\omega + \Gamma_j & -(h_0/n_0)\Gamma_j \end{pmatrix}, \quad (136)$$

where

$$\alpha_{pe} \equiv \left(\frac{\partial P}{\partial e} \right)_n, \quad \alpha_{pn} \equiv \left(\frac{\partial P}{\partial n} \right)_e, \quad (137)$$

$$\alpha_{Te} \equiv \beta_0 h_0 \left(\frac{\partial T}{\partial e} \right)_n, \quad \alpha_{Tn} \equiv \beta_0 h_0 \left(\frac{\partial T}{\partial n} \right)_e, \quad (138)$$

$$\Gamma_{\parallel} = \frac{1}{h_0} \left(\zeta + \frac{4}{3}\eta \right), \quad \Gamma_j = \frac{\beta_0 h_0}{\lambda}. \quad (139)$$

We here neglected the frequency dependence of the thermal conductivity. We can obtain dispersion relations from $\det M = 0$. Those are given as the following to second order in k :

$$\omega \sim -ik^2\Gamma_t, \quad (140)$$

$$\omega \sim -ik^2\Gamma_s \pm kc_s, \quad (141)$$

$$\omega = -ik^2(\eta/h_0), \quad (142)$$

where we introduced the thermal and sound diffusion constants, and the sound velocity:

$$\Gamma_t = \frac{\lambda}{n_0 c_P}, \quad (143)$$

$$\Gamma_s = \frac{1}{2} \left[\Gamma_{\parallel} + \Gamma_t \left(n_0 c_P \left(\frac{T_0 n_0}{h_0} \right)^2 \left(\frac{\partial(\beta\mu)}{\partial n} \right)_e - 1 \right) \right], \quad (144)$$

$$c_s = \left(\frac{\partial P}{\partial e} \right)_{s/n}. \quad (145)$$

Here, c_P is the specific heat at constant pressure. From Eqs. (140)-(142), we see that the equations for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$ do not have unstable modes. Moreover, these dispersions are the same as for the Landau equation [34], i.e., the current density δj^i is irrelevant for slow dynamics.

When does the unstable mode arise in the derivation of the Eckart equation? The unstable mode arises from the expansion about ∂_0 in Eq. (116). Let us now study the modes after the expansion. For simplicity, we consider only the transverse components. Those are given as

$$\begin{pmatrix} -i\omega + k^2(\eta/h_0) & 0 \\ n_0/h_0 & -i\omega - \Gamma_j \end{pmatrix} \begin{pmatrix} \delta \mathbf{p}_{\perp}(\omega, \mathbf{k}) \\ \delta \mathbf{j}_{\perp}(\omega, \mathbf{k}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (146)$$

and dispersions are

$$\omega = -ik^2(\eta/h_0), \quad (147)$$

$$\omega = i\Gamma_j. \quad (148)$$

We see that there are two modes; the first mode is the same as Eq. (142) and the second one is unstable. We note that, before the expansion, the transverse components have the only one mode. Namely, by the ∂_0 expansion, the number of modes increase and the unstable mode arises.

Now, we note that the eigenvalue of the unstable mode is not proportional to the wavenumber k and large. In other words, the unstable mode is massive and not a hydrodynamic mode. Therefore, we can get rid of the unstable mode by introducing a cutoff in the frequency space because hydrodynamics is an effective theory in the low-frequency and -wavenumber region.

VI. SUMMARY

We studied relativistic hydrodynamics in the linear regime with Mori's projection operator method. From the view point of the projection operator method, we discussed that the difference of the frames is not the choices of the reference frames but rather those of the slow variables. We also found that the slow variables for the Landau frame are the conserved charges whereas those for the Eckart frame include the current of the conserved charge, which is not essentially slow. In fact, we derived the slow dynamics by the projection operator method for the sets, $\{\delta e, \delta p^i, \delta n\}$ and $\{\delta e, \delta p^i, \delta n, \delta j^i\}$, as the slow variables. We first showed that the natural choice, Eq. (23), gives the linearized Landau equations. Next, we derived the equations of motion for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$, which include the Landau and Eckart frames in the view point of the projection operator method. And then, we derived the linearized Eckart equation by eliminating δp^i from the above equations.

We also discussed properties of the derived equations. In particular, by considering the Onsager relation, we illustrated that the slow part of the particle current, which is the fluid velocity for the Eckart frame, is determined by Landau's variables, $\{\delta e, \delta p^i, \delta n\}$. Moreover, we studied how the unstable mode of the Eckart equation arises. We found that the ∂_0 expansion for the elimination of δp^i causes the unstable mode. Then, we pointed out that the unstable mode is "massive" and is ruled out by the cutoff in the frequency space. Also, we found that the equations for $\{\delta e, \delta p^i, \delta n, \delta j^i\}$ has the same modes as the Landau equation has. Thus, the particle current δj^i is irrelevant for the first order hydrodynamics. Recently, some authors also point out that the Landau frame is natural for the relativistic hydrodynamics, based on the renormalization group method [32].

Here, we note that this study is first for the derivation of relativistic hydrodynamics based on the projection operator method. We stress that our derivation is independent of the microscopic details; we determine the metric from the Lorentz symmetry on microscopic scale and the thermodynamics. The earlier studies [7–10] assume the relativistic Boltzmann equation as the underlying microscopic theory, which is, however, only valid for a weakly-correlated system like a dilute gas. Furthermore, we note that our study is independent of what are the local equilibrium and the local

rest in the relativistic fluids because we considered the fluctuations from globally equilibrium state. Instead, our study is restricted in the linear regime.

Now, we comment on the Lorentz covariance of linearized hydrodynamics. The linearized Landau equations, (11)-(13), are not Lorentz covariant. The reason is the following: We here considered the fluctuations in the background medium. For such a system, the Lorentz transformation boosts the fluctuations but does not the background. Then, the boosted system differs from that before the boost. Thus, the linearized equations are valid only for the rest frame of the medium and not Lorentz covariant. Actually, by the same reason, the Navier-Stokes equation is Galilei covariant whereas the linearized one is not covariant. However, we note that the linearized Navier-Stokes equation well describes the fluctuation in a fluid [17, 28]. Namely, linearized hydrodynamics has no problem in the linear regime.

In this paper, we used the linear projection operator, and our study is restricted in the linear regime. Then, it is interesting to derive relativistic hydrodynamics by the nonlinear projection operator [29]. We note that, for nonrelativistic fluids, the full Navier-Stokes equation is derived by the nonlinear one [30]. Furthermore, we here focused and discussed the Landau and Eckart frames. Then, we concluded that the Landau frame is natural for the slow dynamics. However, other open problems of relativistic hydrodynamics, such as the acausal propagation, have not been solved yet. It would be interesting to discuss these in the view point of the projection operator method. The projection operator method may give an insight into these problems, as well as that of the frames.

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Appendix A: Properties of inner product

Here, we summarize properties of the inner product defined in Eq. (28). For Hermitian operators,

$$(A(t), B(0)) = (A(t), B(0))^* = (B(0), A(t)), \quad (\text{A1})$$

$$(i\mathcal{L}A(t), B(0)) = -(A(t), i\mathcal{L}B(0)), \quad (\text{A2})$$

$$(i\mathcal{L}A(t), B(0)) = -\frac{i}{\beta} \langle [A(t), B(0)] \rangle_{\text{eq}}, \quad (\text{A3})$$

$$(A(t), B(0)) = \epsilon_A \epsilon_B (A(-t), B(0)) = \epsilon_A \epsilon_B (B(t), A(0)) \quad (\text{A4})$$

are satisfied. Here ϵ_A and ϵ_B denote the sign associated with time reversal transformation, which is defined with the time reveal operator \mathcal{T} as $\mathcal{T}^{-1}A(t)\mathcal{T} = \epsilon_A A(-t)$.

Appendix B: Memory function

In this Appendix, we derive the full expression of the memory function following Ref. [31]. Let us first define the key functions:

$$\dot{\Xi}(t) \equiv (i\mathcal{L}A_n(t), A^m), \quad (\text{B1})$$

$$\ddot{\Xi}(t) \equiv ((i\mathcal{L})^2 A_n(t), A^m). \quad (\text{B2})$$

The reversible term can be expressed by $i\Omega = \dot{\Xi}(0)$. $\ddot{\Xi}(t)$ coincides with the memory function obtained by replacing \mathcal{Q} with unity. Expanding $\exp(t\mathcal{Q}i\mathcal{L})$ in terms of $\mathcal{P}i\mathcal{L}$, we obtain

$$e^{t\mathcal{Q}i\mathcal{L}} = e^{it\mathcal{L}} \left(1 + \sum_{n=1}^{\infty} (-1)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n i\mathcal{L}^P(t_1) \cdots i\mathcal{L}^P(t_n) \right), \quad (\text{B3})$$

where

$$\mathcal{L}^P(t) \equiv e^{-i\mathcal{L}t} \mathcal{P} \mathcal{L} e^{i\mathcal{L}t}. \quad (\text{B4})$$

Acting $\mathcal{L}^P(t)$ to A_n , we obtain

$$i\mathcal{L}^P(t)A_m(-t') = e^{-i\mathcal{L}t}Pi\mathcal{L}e^{i\mathcal{L}t}A_m(-t') = A_n(-t)(i\mathcal{L}A_m(t-t'), A^n) = \dot{\Xi}_m^n(t-t')A_n(-t). \quad (\text{B5})$$

Then, for an operator \mathcal{O} ,

$$\begin{aligned} e^{tQ}i\mathcal{L}\mathcal{O} &= e^{it\mathcal{L}}\left(1 + \sum_{n=1}^{\infty}(-1)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n i\mathcal{L}^P(t_1) \cdots i\mathcal{L}^P(t_n)\right)\mathcal{O} \\ &= \mathcal{O}(t) + \sum_{n=1}^{\infty}(-1)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \\ &\quad \times \dot{\mathcal{O}}_P(t_n)\dot{\Xi}(t_{n-1}-t_n)\dot{\Xi}(t_{n-2}-t_{n-1}) \cdots \dot{\Xi}(t_1-t_2)A(t-t_1) \end{aligned} \quad (\text{B6})$$

is satisfied. Here we defined

$$\dot{\mathcal{O}}_P(t) = (i\mathcal{L}\mathcal{O}(t), A^m). \quad (\text{B7})$$

Performing the Laplace transform, we obtain

$$\int dt e^{-tz} e^{tQ}i\mathcal{L}\mathcal{O} = \mathcal{O}(z) - \dot{\mathcal{O}}_P(z) \frac{1}{1 + \dot{\Xi}(z)} A(z). \quad (\text{B8})$$

Using this equation, we find the full expressions of the nose term and the memory function in the Fourier-Laplace space:

$$R(z, \mathbf{k}) = i\mathcal{L}A(z, \mathbf{k}) - (i\Omega + \ddot{\Xi}(z, \mathbf{k})) \frac{1}{1 + \dot{\Xi}(z, \mathbf{k})} A(z, \mathbf{k}), \quad (\text{B9})$$

$$\Phi(z, \mathbf{k}) = -(\ddot{\Xi}(z, \mathbf{k}) - i\Omega(\mathbf{k})\dot{\Xi}(z, \mathbf{k})) \frac{1}{1 + \dot{\Xi}(z, \mathbf{k})}. \quad (\text{B10})$$

Appendix C: Projection of $\delta\mathbf{j}$ on $\{\delta e, \delta\mathbf{p}, \delta n\}$

Here, we show that the projection of $\delta\mathbf{j}$ on $\{\delta n, \delta e, \delta\mathbf{p}\}$ gives Eq. (128). Namely, we will show

$$\mathcal{P}\left[\delta\mathbf{j}(t, \mathbf{x}) - \frac{n_0}{h_0}\delta\mathbf{p}(t, \mathbf{x}) + \lambda\left(\frac{n_0 T_0}{h_0}\right)^2 \nabla\delta(\beta\mu)(t, \mathbf{x})\right] = 0, \quad (\text{C1})$$

to the first order in k . In the Fourier-Laplace space, Eq. (C1) becomes

$$\mathcal{P}\left[\delta\mathbf{j}(z, \mathbf{k}) - \frac{n_0}{h_0}\delta\mathbf{p}(z, \mathbf{k}) + i\mathbf{k}\lambda\left(\frac{n_0 T_0}{h_0}\right)^2 \delta(\beta\mu)(z, \mathbf{k})\right] = 0. \quad (\text{C2})$$

Here we decompose Eq. (C2) into the longitudinal and transverse components:

$$\mathcal{P}\left[\delta j_{\parallel}(z, \mathbf{k}) - \frac{n_0}{h_0}\delta p_{\parallel}(z, \mathbf{k}) + ik\lambda\left(\frac{n_0 T_0}{h_0}\right)^2 \delta(\beta\mu)(z, \mathbf{k})\right] = 0, \quad (\text{C3})$$

$$\mathcal{P}\left[\delta\mathbf{j}_{\perp}(z, \mathbf{k}) - \frac{n_0}{h_0}\delta\mathbf{p}_{\perp}(z, \mathbf{k})\right] = 0. \quad (\text{C4})$$

Let us show the equation for the transverse component, Eq. (C4). Consider the projections, $\mathcal{P}\delta\mathbf{j}_{\perp}(z, \mathbf{k})$ and $\mathcal{P}\delta\mathbf{p}_{\perp}(z, \mathbf{k})$, which are given as

$$\mathcal{P}\delta\mathbf{j}_{\perp}(z, \mathbf{k}) = \Xi_{j_{\perp}}^n(z, \mathbf{k})\delta n(0, \mathbf{k}) + \Xi_{j_{\perp}}^e(z, \mathbf{k})\delta e(0, \mathbf{k}) + \Xi_{j_{\perp}}^{p_{\perp}}(z, \mathbf{k})\delta\mathbf{p}_{\perp}(0, \mathbf{k}), \quad (\text{C5})$$

$$\mathcal{P}\delta\mathbf{p}_{\perp}(z, \mathbf{k}) = \Xi_{p_{\perp}}^n(z, \mathbf{k})\delta n(0, \mathbf{k}) + \Xi_{p_{\perp}}^e(z, \mathbf{k})\delta e(0, \mathbf{k}) + \Xi_{p_{\perp}}^{p_{\perp}}(z, \mathbf{k})\delta\mathbf{p}_{\perp}(0, \mathbf{k}). \quad (\text{C6})$$

We note that \mathbf{k} expansions of $\Xi_{j_{\perp}}^n(z, \mathbf{k})$ and $\Xi_{j_{\perp}}^e(z, \mathbf{k})$ give only odd-order terms in \mathbf{k} from the tensor structure. Then, we can drop $\Xi_{j_{\perp}}^n(z, \mathbf{k})$ and $\Xi_{j_{\perp}}^e(z, \mathbf{k})$ because the odd terms are orthogonal to the transverse component. Then, Eq (C4) turns out to be

$$\left[\Xi_{j_{\perp}}^{p_{\perp}}(z, \mathbf{k}) - \frac{n_0}{h_0}\Xi_{p_{\perp}}^{p_{\perp}}(z, \mathbf{k})\right]\delta\mathbf{p}_{\perp}(0, \mathbf{k}) = 0. \quad (\text{C7})$$

Here, let us consider $\Xi_{j\perp}^{p\perp}(z, \mathbf{k})$ and $\Xi_{p\perp}^{p\perp}(z, \mathbf{k})$. For this task, we now use the equations of motion for $\{\delta n, \delta e, \delta \mathbf{p}, \delta \mathbf{j}\}$, Eqs. (97), (98), (106) and (113). From these equations, we obtain the equations for the transverse components as

$$\begin{pmatrix} z + k^2 \Gamma_\perp & 0 \\ z + \Gamma_j & -(h_0/n_0)\Gamma_j \end{pmatrix} \begin{pmatrix} \delta \mathbf{p}_\perp(z, \mathbf{k}) \\ \delta \mathbf{j}_\perp(z, \mathbf{k}) \end{pmatrix} = \begin{pmatrix} \delta \mathbf{p}_\perp(t=0, \mathbf{k}) \\ \delta \mathbf{p}_\perp(t=0, \mathbf{k}) + n_0 T_0 g^{jj} \left(\delta \mathbf{j}_\perp(t=0, \mathbf{k}) - \frac{n_0}{h_0} \delta \mathbf{p}_\perp(t=0, \mathbf{k}) \right) \end{pmatrix}. \quad (\text{C8})$$

Therefore,

$$\delta \mathbf{p}_\perp(z, \mathbf{k}) = \frac{1}{z + k^2 \Gamma_\perp} \delta \mathbf{p}_\perp(t=0, \mathbf{k}), \quad (\text{C9})$$

$$\delta \mathbf{j}_\perp(z, \mathbf{k}) = \frac{(n_0/h_0)}{z + k^2 \Gamma_\perp} \delta \mathbf{p}_\perp(t=0, \mathbf{k}) - \frac{n_0^2 T_0 g^{jj}}{h_0 \Gamma_j} \left(\delta \mathbf{j}_\perp(t=0, \mathbf{k}) - \frac{n_0}{h_0} \delta \mathbf{p}_\perp(t=0, \mathbf{k}) \right). \quad (\text{C10})$$

If we notice that the conjugate variable for δp^i is

$$A^{p^i} = \frac{1}{h_0 T_0} \delta p^i, \quad (\text{C11})$$

in the space of $\{\delta e, \delta \mathbf{p}, \delta n\}$, we find

$$\Xi_{j\perp}^{p\perp}(z, \mathbf{k}) = \frac{(n_0/h_0)}{z + k^2 \Gamma_\perp}, \quad (\text{C12})$$

$$\Xi_{p\perp}^{p\perp}(z, \mathbf{k}) = \frac{1}{z + k^2 \Gamma_\perp}, \quad (\text{C13})$$

where we used Eqs. (67) and (68). Finally, we arrive at

$$\left[\Xi_{j\perp}^{p\perp}(z, \mathbf{k}) - \frac{n_0}{h_0} \Xi_{p\perp}^{p\perp}(z, \mathbf{k}) \right] = 0. \quad (\text{C14})$$

We have shown the equation for the longitudinal component, Eq. (C4). By the similar procedure, we can show that for the longitudinal component, Eq. (C3).

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